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Approximation of the soil particle-size distribution curve using a NURBS curve

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Abstract: Soil particle-size distribution or soil texture presents one of the most important physical properties. There are various systems of the classification systems for soil particle-size fractions with different boundaries. Our effort was concentrated on the mathematical approach to evaluate the existing data and convert it to the form of a reconstructed cumulative particle-size curve which will allow reading concentration of any desired particle size. Non-Uniform Rational B-Splines (NURBS) curves therefore represent a generalization of B-splines and Bézier curves by extending the definition by an element of rationality, which is represented by the weights of the control points, and a nodal vector of parametrization, which represents the element of uniformity. The NURBS curve was used for smooth (depending on the degree of the curve used) and as tight as possible approximation of the arranged control points, the connecting lines of which forms a convex envelope for its individual parts. The NURBS approximation curve is therefore determined by the ordered control points and their connecting lines, the weights of these points, the degree of the curve and the nodal vector of parametrization. However, the construction of the approximation curve is primarily dependent on a limited number of points of the experimentally determined particle-size distribution curves, and for curves with significant breaks in the course, one must consider either a lower accuracy of the approximation or the necessity of "improving" the approximation using the weights of individual points, inserting additional points or working with a nodal vector of parametrization. For basic approximation, the PUGIS system (Czech soil information system) offers automatic approximation using all variants mentioned in the article as well as the possibility of individual changes in the weights of control points, in their number and position, and in the nodal vector of parametrization.

Keywords: data harmonization; legacy soil database; mathematical approach; soil particles cumulative curve restoration; soil texture

Soil consists of an assemblage of particles that differ widely in size and shape. Soil texture is considered as one of the most important features of soil. The term soil texture is an expression of the dominant particle sizes, or the proportion of particle-size fractions found in soil and has both qualitative and quantitative

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Table 1. Classification and characterization of the particle-size fractions (Jahn et al. 2006)

Equivalent diameter		Denomination		Symbol	Basic division
(mm)	(µm)				
≥ 200		boulders			
200–63		stones			
63–20		coarse gravel			coarse fraction
20–6.3		medium gravel			
6.3–2		fine gravel			
	2 000–630	coarse sand			
2–0.063	630–200	medium sand	sand	S	
	200–63	fine sand			
	63–20	coarse silt			
0.063–0.002	20–6.3	medium silt	silt	Si	fine earth fraction
	6.3–2.0	fine silt			
	2.0–0.63	coarse clay			
< 0.002	0.63–0.2	medium clay	clay	C	
	< 0.2	fine clay			

connotations. The classification of the particle-size fractions is shown in Table 1 in which the system suggested and used by Jahn et al. (2006) is given.

Particle-size fractions are measured in intervals of orders of magnitude with the steps from 2 to 6.3 and 20, as the very large particles size ranges from > 200 mm to < 0.0002 mm can only be represented logarithmically, and 6.3 divides the range of 2–30 on the logarithmic scale in equal parts (Blum et al. 2018). Nevertheless, there are other systems for particle-size fraction classification with different boundaries, for example USDA Soil Taxonomy (Soil Survey Staff 1999).

Cumulative particle-size distribution curves are used very often for textural composition description. Examples of such curves are given in Figure 1 (Blume et al. 2016). There are many methods for determination of soil texture. A very good overview of them was published by Pansu and Gautheryou (2003). Morais et al. (2019) published an innovative contribution based on predicting soil texture using image analysis.

Frequently, there is a problem based on the evaluation of large databases of legacy data where different determination methods and different classification systems were used. Zádorová et al. (2018, 2020) tried to solve that problem. The authors tested several models, finally a simple logarithmic-linear transformation appeared to provide the best results. Previously there

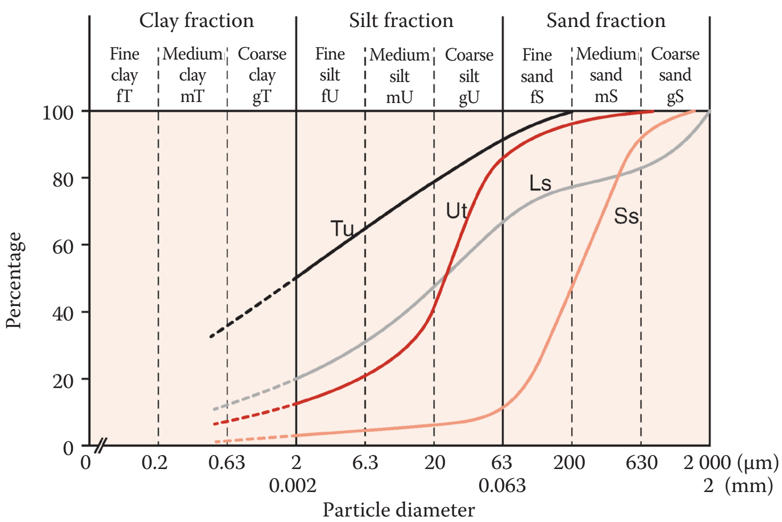

Figure 1. Cumulative particle-size distribution curves for fine earth for several soils, namely sand (Ss), loess (Ut), glacial loam (Ls), and clay rich mud (Tu) (Blume et al. 2016)

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was also an attempt published by Němeček et al. (2011) based on a simple linear regression between clay fractions with different boundaries. The aim of this paper is to show a solution to this problem using the NURBS (Non-Uniform Rational B-Splines) curves (Piegl & Tiller 1997).

METHODOLOGY

Our effort was concentrated on the mathematical approach to evaluate the existing data and convert it to a reconstructed cumulative particle-size distribution curve which will allow the read the concentration of any desired particle size, using the NURBS curve with some modifications.

In the mathematical part of our work the publication by Piegl and Tiller (1997) was used.

Theory

NURBS curve and its construction

NURBS curves represent a generalization of B-splines and Bézier curves by extending the definition by an element of rationality, which is represented by the weights of the control points, and a nodal vector of parametrization, which represents the element of uniformity. The NURBS curve is used for smooth (depending on the degree of the curve used) and as tight as possible approximation of the arranged control points, the connecting lines of which forms a convex envelope for its individual parts.

The NURBS approximation curve is therefore determined by the ordered control points and their connecting lines, the weights of these points, the degree of the curve and the nodal vector of parametrization.

Definition of a NURBS curve

For:

$m + 1$ control points P,

$m + 1$ of positive real numbers w called weights,

the degree of the curve n,

curve order $k = n + 1$

and nodal vector of parametrization t = $(t_0, t_1, ..., t_{n+m+1})$

A NURBS curve is defined by the equation:

$$C(t) = \frac{\sum_{i=0}^{m} w_i P_i N_i^n(t)}{\sum_{i=0}^{m} w_i N_i^n(t)} \qquad (1)$$

where:

$t \in \langle t_n, t_{m+1} \rangle$

The basis function N_i^n is defined for the node vector as follows:

$$N_i^0(t) = \begin{cases} 1 \text{ pro } t \in \langle t_i, t_{i+1}) \\ 0 \text{ pro } t \notin \langle t_i, t_{i+1}) \end{cases} \qquad (2)$$

$$N_i^n(t) = \frac{t - t_i}{t_{i+n} - t_i} N_i^{n-1}(t) + \frac{t_{i+n+1} - t}{t_{i+n+1} - t_{i+1}} N_{i+1}^{n-1}(t)$$

The parametrization vector is represented by any non-decreasing sequence of values satisfying the following conditions.

(1) The sequence of values in the nodal vector of parametrization must be non-decreasing. Therefore, for the values of the components of the nodal vector of parametrization:

$$t_i \le t_{i+1} \qquad (3)$$

(2) The number of values of the nodal vector of parametrization u is equal to the sum of the number of control points $m + 1$ and the degree n of the curve increased by one.

$$u = m + 1 + n + 1 \qquad (4)$$

It was called a nodal vector of parametrization uniform if the intervals between the individual components of the vector are the same. If the intervals are not the same, it is a non-uniform vector.

To approximate the particle-size distribution curve, it was required that the resulting approximation curve passes through the extreme control points. For achieving this, in the case of a uniform B-spline curve, by inserting the same n endpoints at the beginning and end of the sequence of control points. For a non-uniform B-spline curve, it could achieve the same effect by modifying the nodal vector of parametrization by inserting multiple $n + 1$ nodes.

To simplify the design of the approximation curve, it was considered the unit weights of all control points of the control polygon and work with a uniform nodal vector of parametrization. Given that the resulting approximation curve may not, in this standard setting, meet the requirements of the solver, it is necessary to consider the possibilities of influencing its shape.

The shape of the resulting approximation curve is generally determined by the chosen degree of the curve n, the weights of individual control points w_i and the distances $t_{i+1} - t_i$ in the nodal vector of parametrization. To simplify the design of the weights of individual control points or distances in the nodal

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vector of parametrization, it could be used geometric methods that are inspired by minimizing the sum of the distances of the points of the approximation curve from a given fixed point. In this case, the fixed points are the centers of gravity of the partial control polygons. It was called the method of determining the weights of control points or determining distances within the nodal vector of parametrization, the center of gravity method.

Center of gravity method for determining the weights of control points

For the control polygon defined by the control points $(P_i)_{i=0}^{m}$, the centroids of individual partial control polygons T_i are defined by the equation:

$$T_i = \frac{1}{n+1}\sum_{j=i}^{i+n} P_j \text{ , } i = 0, \ldots, m-n \tag{5}$$

The centroid weights of the individual control points forming the control polygon are defined as follows:

$$w_i = \frac{1}{i+1}\sum_{j=0}^{i}\left|P_i T_j\right| \text{ , } i < n \tag{6}$$

$$w_i = \frac{1}{n+1}\sum_{j=i-n}^{i}\left|P_i T_j\right| \text{ , } n \le i \le m-n \tag{7}$$

$$w_i = \frac{1}{m-i+1}\sum_{j=i-n}^{m-n}\left|P_i T_j\right| \text{, } i > m-n \tag{8}$$

The centroid weight w_i of the control point P_i is therefore equal to the average distance of the control point from the centroids of all partial control polygons in which the point P_i participates in the construction of the approximation curve.

Center of gravity method for nodal vector of parametrization determination

In addition to determining the weights of individual control points, the shape of the resulting approximation curve can be influenced by the design of the nodal vector of parametrization, which is again based on the geometric properties of the control polygon defined by the control points $(P_i)_{i=0}^{m}$.

This method is based on the position of the centroids of partial control polygons:

$$\left(P_j\right)_{j=i-1}^{i+n}\text{, } i = 1, \ldots, m-n \tag{9}$$

With $n + 2$ control points, which define two adjacent segments of the curve connected in node t_i of nodal vector of parametrization and thus considers the position of control points. The nodal vector of parametrization, determined by the center of gravity method, is called the center of gravity nodal vector of parametrization.

For the control polygon defined by the control points $(P_i)_{i=0}^{m}$ and the degree of the curve n, the centroids of the individual sub-polygons T_i are defined by the equations:

$$T_0 = P_0$$

$$T_i = \frac{1}{n+2}\sum_{j=i-1}^{i+n} P_j \text{ , } i = 1, \ldots, m-n \tag{10}$$

$$T_{m-n+1} = P_m$$

The distances of successive centroids are then defined by the equation:

$$l_i = \left|T_{i-1}T_i\right|,\ i = 1, \ldots, m-n+1$$

and the sum of all centroid distances:

$$L = \sum_{i=1}^{m-n+1} l_i \tag{11}$$

The center of gravity nodal vector of parametrization for the approximation curve is then given by the equations:

$$t_i = 0,\ i = 0, \ldots, n$$

$$t_i = \frac{1}{L}\sum_{j=1}^{i-n} l_j \text{ , } i = n+1, \ldots, m-n-1 \tag{12}$$

$$t_i = 1,\ i = m-n, \ldots, m$$

The algorithm for calculation of basis B-spline functions in PL/SQL (Oracle database procedural language) is presented in the Supplementary Material 1 in Electronic Supplementary Material (ESM).

Examples of B-spline basis functions for different degrees of NURBS

Examples of B-spline basis functions for different degrees of NURBS are shown in Figure 2. Curves were calculated as follows.

Number of control points: $m + 1 = 5$

Curve order $k = 2$ (for linear B-spline) and nodal vector of parametrization $t = (0, 0, 1, 2, 3, 4, 4)$

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Curve order k = 3 (quadratic B-spline) and nodal vector of parametrization t = (0, 0, 0, 1, 2, 3, 3, 3)

Curve order k = 4 (cubic B-spline) and nodal vector of parametrization t = (0, 0, 0, 0, 1, 2, 2, 2, 2)

Application

Approximation of the particle-size distribution curve by the NURBS curve

The particle-size distribution curve expresses the cumulative relative proportion of individual particle size fractions, given by their share in the total mass of the soil sample. The curve is given for the particle size in a logarithmic scale, which allows a more detailed view of the particle sizes for fine fractions that significantly influence soil properties.

For the approximation of the particle-size distribution curve, it was started with the logarithmic display of the particle-size distribution curve. The sorted nodal points of the curve stored in the database as a control sequence (control points) were used. The sequence of control points forms a convex envelope of the resulting approximate NURBS curve, which we will use with advantage to ensure the character of the particle-size distribution curve as a summation line. The approximation curve passes through the extreme points of the sequence (secured by the construction of the nodal vector of parametrization) and the approximation curve approaches the "inner" points of the particle-size distribution curve depending on the slope ratios between adjacent sections of the particle-size distribution curve. For closer approximation, the resulting curve must be "pulled" to the original particle-size distribution curve. This can be achieved either by designing a non-equidistant nodal vector of para-

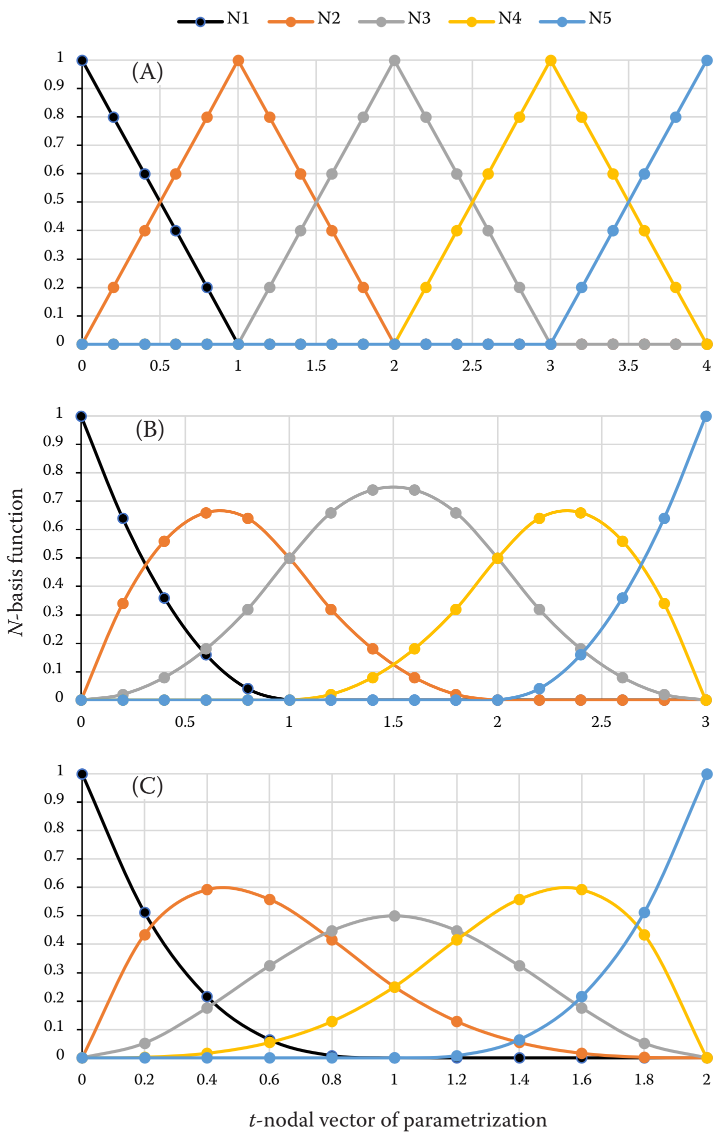

Figure 2. B-spline basis functions of the: 1st degree and nodal vector of parametrization t = (0,0,1,2,3,4,4) (A), 2nd degree and nodal vector of parametrization t = (0,0,0,1,2,3,3,3) (B), 3rd degree and nodal vector of parametrization t = (0,0,0,0,1,2,2,2,2) (C)

N1, N2, N3, N4, and N5 – No. of points

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Table 2. Experimentally determined points of the particle-size distribution curves A (Examples 1–4) and B (Examples 5–8).

A particle-size distribution			B particle-size distribution		
d (mm)	p (%)	cumulative p (%)	d (mm)	p (%)	cumulative p (%)
0.001	45.40	45.40	0.002	22.50	22.50
0.010	31.40	76.80	0.010	0.00	22.50
0.050	4.00	80.80	0.050	10.90	33.40
0.250	8.40	89.20	0.250	38.70	72.10
2.000	10.80	100.00	2.000	27.90	100.00

d – particle size; p – percentage of fraction

metrization, by weighting individual control points or by inserting intermediate values into the sequence of control points. To approximate the particle-size distribution curve, it was used the experimentally determined points of the curve, where the particle size d is considered on a logarithmic scale.

To test different approaches to approximate particle-size distribution curves, two measured particle-size distribution curves were used (Table 2). Datapoints of curve A were used for following Examples 1–4, and datapoints of curve B were used for Examples 5–8. Various scenarios, i.e., Examples, are discussed together with their results in the following part.

RESULTS AND DISCUSSION

Example 1 – Control points from particle-size distribution curve in logarithmic scale with uniform weight

Control points P_i for A particle-size distribution curve and their weights (for simplicity, we consider the same weight for all control points: $w = 1$) are presented in Table 3.

Uniform nodal vector of parametrization

To simplify the example, a uniform nodal vector of parametrization was considered.

For the individual degrees of the approximation curve considered (where we proceed from the condition that the degree of the approximation curve must be smaller than the number of sections of the approximated polygon) $n < m$, we get the following parametrization vectors:

For a curve of the 1st degree (2nd order) – (linear B-spline) and a sequence of 5 control points:

$$t = (-3, -3, -2.1747, -1.3495, -0.5242, 0.3010, 0.3010)$$

For a curve of the 2nd degree (3rd order) – (quadratic B-spline) and a sequence of 5 control points:

$$t = (-3, -3, -3, -1.8997, -0.7993, 0.3010, 0.3010, 0.3010)$$

For a curve of the 3rd degree (4th order) – (cubic B-spline) and a sequence of 5 control points:

$$t = (-3, -3, -3, -3, -1.3495, 0.3010, 0.3010, 0.3010, 0.3010)$$

Resulting approximation curves are shown together with the control point in Figure 3. Points of the approximation curves are also shown in Table S1 in ESM. Differences between the control points and the points of the approximation curves are shown in Table 3. The difference between the points of the

Table 3. Control points (P_i), their weights, and differences between the control points and the points of the approximation curves (3rd, 2nd, and 1st degree) obtained for Example 1 that assumed control points from the particle-size distribution curve in logarithmic scale with the uniform nodal vector of parametrization

P_i	$X = \log(d)$	Y = cumulative p (%)	Weight	3rd degree curve	2nd degree curve	1st degree curve
				(%)		
P_0	log(0.001)	45.40	1	0	0	0
P_1	log(0.01)	76.80	1	4.42	3.59	0.38
P_2	log(0.05)	80.80	1	1.09	0.54	0.11
P_3	log(0.25)	89.20	1	0.06	0.03	0
P_4	log(2)	100.00	1	0	0	0

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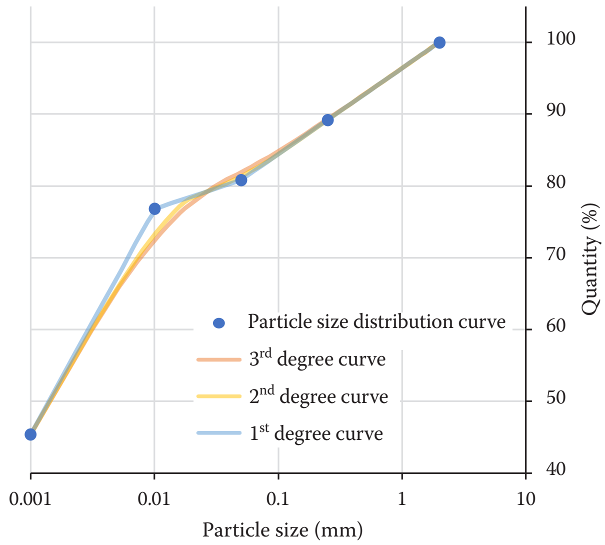

Figure 3. Control points and approximation curves for Example 1 that assumed control points from the particle-size distribution curve in logarithmic scale with the uniform nodal vector of parametrization

approximation curves and the points of the particle-size distribution curve is therefore 4.42% for the cubic curve, 3.59% for the quadratic curve. We consider the linear function only for comparison. If there is a need to use a linear function, approximation can be achieved more quickly using a simple linear interpolation.

Example 2 – Control points from particle-size distribution curve in logarithmic scale with uniform weight and nodal vector of parametrization determined by center of gravity method

Determination of parametrization vector using center of gravity method

For the individual degrees of the approximation curve under consideration (where we proceed from the condition that the degree of the approximation curve must be smaller than the number of sections of the approximated polygon) $n < m$, we calculate the center of gravity of the partial control polygons. The equations for calculating the centroids of individual partial polygons are as follows:

$$T_0 = P_0$$

$$T_i = \frac{1}{n+2}\sum_{j=i-1}^{i+n} P_j\ ,\ i = 1, \ldots, m-n \qquad (13)$$

$$T_{m-n+1} = P_m$$

For the example with 5 control points $m = 4$ of the control polygon:

For a curve of the 1st degree ($n = 1$) (2nd order) – (linear B-spline)

$$T_0 = P_0 = (\log(0.001), 45.4)$$

$$T_1 = 1/3(P_0 + P_1 + P_2) = 1/3[(\log(0.001), 45.4) + (\log(0.01), 76.8) + (\log(0.05), 80.80)] = (\log(0.00794), 67.67)$$

$$T_2 = 1/3(P_1 + P_2 + P_3) = 1/3[(\log(0.01), 76.8) + (\log(0.05), 80.80) + (\log(0.25), 89.20)] = (\log(0.05), 82.27)$$

$$T_3 = 1/3(P_2 + P_3 + P_4) = 1/3[(\log(0.05), 80.80) + (\log(0.25), 89.20) + (\log(2), 100)] = (\log(0.2924), 90)$$

$$T_4 = P_4 = (\log(2), 100)$$

The distances of successive centroids are then defined by the equation:

$$l_i = |T_{i-1}T_i|,\ i = 1, \ldots, m-n+1$$

$$l_1 = |T_0T_1| = \sqrt[2]{(\log(0.00794) - \log(0.001))^2 + (67.67 - 45.4)^2} = 22.29$$

$$l_2 = |T_1T_2| = \sqrt[2]{(\log(0.05) - \log(0.00794))^2 + (82.27 - 67.67)^2} = 14.62$$

$$l_3 = |T_2T_3| = \sqrt[2]{(\log(0.2924) - \log(0.05))^2 + (90 - 82.27)^2} = 7.77$$

$$l_4 = |T_3T_4| = \sqrt[2]{(\log(2) - \log(0.2924))^2 + (100 - 90)^2} = 10.03$$

and the sum of all centroid distances:

$$L = \sum_{i=1}^{4} l_i = (22.29 + 14.62 + 7.77 + 10.03) = 54.71$$

The center of gravity nodal vector of parametrization for the approximation curve is given by the equations:

$$t_i = 0,\ i = 0, \ldots, n$$

$$t_i = \frac{1}{L}\sum_{j=1}^{i-n} l_j\ ,\ i = \mathrm{n} + 1, \ldots, m-n-1 \qquad (14)$$

$$t_i = 1,\ i = m-n, \ldots, m$$

$$t_0 = 0,\ t_1 = 0$$

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$$t_2 = \frac{1}{54.71} 22.29 = 0.407,$$

$$t_3 = \frac{1}{54.71}(22.29 + 14.62) = 0.675,$$

$$t_4 = \frac{1}{54.71}(22.29 + 14.62 + 7.77) = 0.817,$$

$T_5 = 1$, $t_6 = 1$

For a curve of the 2nd degree ($n = 2$) (3rd order) – (quadratic B-spline)

$T_0 = P_0 = (\log(0.001), 45.4)$

$T_1 = 1/4(P_0 + P_1 + P_2 + P_3) = 1/4[(\log(0.001), 45.4) + (\log(0.01), 76.8) + (\log(0.05), 80.80) + (\log(0.25), 89.20)] = (\log(0.0188), 73.05)$

$T_2 = 1/4(P_1 + P_2 + P_3 + P_4) = 1/4[(\log(0.01), 76.8) + (\log(0.05), 80.80) + (\log(0.25), 89.20) + (\log(2), 100)] = (\log(0.12574), 86.7)$

$T_3 = P_4 = (\log(2), 100)$

The distances of successive centroids are then defined by the equation:

$l_i = |T_{i-1}T_i|$, $i = 1, ..., m - n + 1$

$$l_1 = |T_0T_1| = \sqrt[2]{(\log(0.0188) - \log(0.001))^2 + (73.05 - 45.4)^2} = 27.68$$

$$l_2 = |T_1T_2| = \sqrt[2]{(\log(0.12574) - \log(0.0188))^2 + (86.7 - 73.05)^2} = 13.68$$

$$l_3 = |T_2T_3| = \sqrt[2]{(\log(2) - \log(0.12574))^2 + (100 - 86.7)^2} = 13.35$$

and the sum of all centroid distances:

$$L = \sum_{i=1}^{3} l_i = (27.68 + 13.68 + 13.35) = 54.71$$

The center of gravity nodal vector of parametrization for the approximation curve is given by the equations:

$t_i = 0$, $i = 0, ..., n$

$$t_i = \frac{1}{L}\sum_{j=1}^{i-n} l_j \text{ , } i = \mathrm{n} + 1, ..., m - n - 1 \tag{15}$$

$t_i = 1$, $i = m - n, ..., m$

$t_0 = 0$, $t_1 = 0$, $t_2 = 0$

$$t_3 = \frac{1}{54.71} 27.68 = 0.506,$$

$$t_4 = \frac{1}{54.71}(27.68 + 13.68) = 0.756,$$

$t_5 = 1$, $t_6 = 1$, $t_7 = 1$

For a curve of the 3rd degree ($n = 3$) (4th order) – (cubic B-spline)

$T_0 = P_0 = (\log(0.001), 45.4)$

$T_1 = 1/5(P_0 + P_1 + P_2 + P_3 + P_4) = 1/5[(\log(0.001), 45.4) + (\log(0.01), 76.8) + (\log(0.05), 80.80) + (\log(0.25), 89.20) + (\log(2), 100)] = (\log(0.04782), 78.44)$

$T_2 = P_4 = (\log(2), 100)$

The distances of successive centroids are then defined by the equation:

$l_i = |T_{i-1}T_i|$, $i = 1, ..., m - n + 1$

$$l_1 = |T_0T_1| = \sqrt[2]{(\log(0.04782) - \log(0.001))^2 + (78.44 - 45.4)^2} = 33.08$$

$$l_2 = |T_1T_2| = \sqrt[2]{(\log(2) - \log(0.04782))^2 + (100 - 78.44)^2} = 21.62$$

and the sum of all centroid distances:

$$L = \sum_{i=1}^{2} l_i = (33.08 + 21.62) = 54.70$$

The center of gravity nodal vector of parametrization for the approximation curve is given by the equations:

$t_i = 0$, $i = 0, ..., n$

$$t_i = \frac{1}{L}\sum_{j=1}^{i-n} l_j \text{ , } i = \mathrm{n} + 1, ..., m - n - 1$$

$t_i = 1$, $i = m - n, ..., m$

$t_0 = 0$, $t_1 = 0$, $t_2 = 0$, , $t_3 = 0$,

$$t_4 = \frac{1}{54.70} 33.08 = 0.605,$$

$t_5 = 1$, $t_6 = 1$, $t_7 = 1$, $t_8 = 1$

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Table 4. Summary table of centroids of partial control polygons for degrees 1, 2 and 3 of the approximation curve

3rd degree		2nd degree		1st degree	
d (mm)	cumulative *p* (%)	*d* (mm)	cumulative *p* (%)	*d* (mm)	cumulative *p* (%)
0.0010	45.40	0.0010	45.40	0.0010	45.40
0.0478	78.44	0.0188	73.05	0.0079	67.67
2.0000	100.00	0.1257	86.70	0.0500	82.27
		2.0000	100.00	0.2924	90.00
				2.0000	100.00

d – particle size; *p* – percentage of fraction

Centroids of partial control polygons for degrees 1, 2 and 3 of the approximation curves are presented in Table 4.

After converting the calculated nodes of nodal vectors of parametrization to a logarithmic scale, for a particle size of 0.001 mm to 2 mm, we get:

For a curve of the 1st degree (2nd order) – (linear B-spline) and a sequence of 5 control points:

t = (–3.0, –3.0, –1.6555, –0.7733, –0.3044, 0.3010, 0.3010)

For a curve of the 2nd degree (3rd order) – (quadratic B-spline) and a sequence of 5 control points:

t = (–3.0, –3.0, –3.0, –1.3299, –0.5047, 0.3010, 0.3010, 0.3010)

For a curve of the 3rd degree (4th order) – (cubic B-spline) and a sequence of 5 control points:

t = (–3.0, –3.0, –3.0, –3.0, –1.0037, 0.3010, 0.3010, 0.3010, 0.3010)

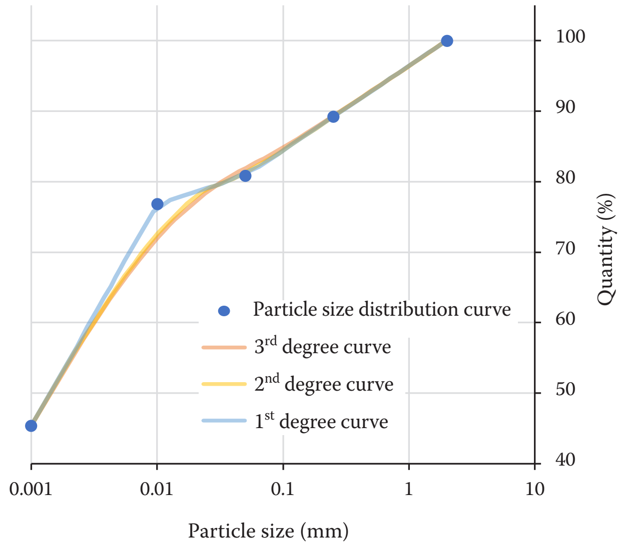

Figure 4. Control points and approximation curves for Example 2 that assumed control points from the particle-size distribution curve in logarithmic scale with the non-uniform nodal vector of parametrization determined by the center of gravity method

Resulting approximation curves are shown together with the control points of A particle-size distribution curve in Figure 4. Points of the approximation curves are also shown in Table S2 in ESM. Differences

Table 5. Control points (P_i), their weights, and differences between the control points and the points of the approximation curves (3rd, 2nd, and 1st degree) obtained for Example 2 that assumed control points from the particle-size distribution curve in logarithmic scale with the non-uniform nodal vector of parametrization determined by the center of gravity method

P_i	X = log(d)	Y = cumulative p (%)	Weight	3rd degree curve (%)	2nd degree curve (%)	1st degree curve (%)
P_0	log(0.001)	45.40	1	0	0	0
P_1	log(0.01)	76.80	1	4.88	4.27	0.68
P_2	log(0.05)	80.80	1	1.08	0.42	0.24
P_3	log(0.25)	89.20	1	0.05	0.04	0.01
P_4	log(2)	100.00	1	0	0	0

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between the control points and the points of the approximation curves are shown in Table 5. The difference between the points of the approximation curves and the points of the particle-size distribution curve is therefore 4.88% for the cubic curve, 4.27% for the quadratic curve.

The effect of using a non-uniform nodal vector of parametrization determined by the center of gravity method is negligible, and in many cases the course of the approximation curves tends to deteriorate. To significantly improve the approximation, it is therefore necessary to look for other alternative solutions.

Example 3 – Inserting of intermediate values in the sequence of control points

One of the alternatives for a closer approximation of the particle-size distribution curves is to insert intermediate values in the sequence of control points. For embedded values on the summation line, the logarithms of the particle size was used. Intermediate points of the particle-size distribution curve are determined by linear interpolation. For approximation, we then have a sequence of 8 points of the particle-size distribution curve A.

Control points P_i for A particle-size distribution curve and their weights (for simplicity, we consider, for original particle-size distribution curve values, uniform weight $w = 1$ and, for embedded values uniform weight $w = 0.5$) are presented in Table 6.

Uniform nodal vector of parametrization

To simplify the example, first was considered a uniform parametrization vector.

For the individual degrees of the curve considered (where we proceed from the condition that the degree of the approximation curve must be smaller than the number of sections of the approximated polygon) $n < m$, we get the following parametrization vectors:

For a curve of the 1st degree (2nd order) – (linear B-spline) and a sequence of 8 control points:

$$t = (-3, -3, -2.5284, -2.0569, -1.5853, -1.1137, -0.6421, -0.1706, 0.3010, 0.3010)$$

where the initial 2 nodes correspond to log(0.001), the final 2 nodes correspond to log(2), and the remaining 6 nodes equidistantly divide the intermediate interval.

For a curve of 2nd degree (3rd order) – (quadratic B-spline) and a sequence of 8 control points:

$$t = (-3, -3, -3, -2.4498, -1.8997, -1.3495, -0.7993, -0.2491, 0.3010, 0.3010, 0.3010)$$

where the initial 3 nodes correspond to log(0.001), the final 3 nodes correspond to log(2), and the remaining 5 nodes equidistantly divide the intermediate interval.

For a curve of 3rd degree (4th order) – (cubic B-spline) and a sequence of 8 control points:

$$t = (-3, -3, -3, -3, -2.3398, -1.6796, -1.0194, -0.3592, 0.3010, 0.3010, 0.3010, 0.3010)$$

where the initial 4 nodes correspond to log(0.001), the final 4 nodes correspond to log(2), and the remaining 4 nodes equidistantly divide the intermediate interval.

Resulting approximation curves are shown together with the control points in Figure 5. Points of the

Table 6. Control points (P_i), their weights, and differences between the control points and the points of the approximation curves obtained for Example 3 that assumed the uniform nodal vector of parametrization and particle-size distribution curve with embedded intermediate values for particle diameter d = 0.0032, 0.0224, and 0.1118 mm by linear interpolation

P_i	$X = \log(d)$	Y = cumulative p (%)	Weight	3rd degree curve	2nd degree curve	1st degree curve
					(%)	
P_0	log(0.001)	45.40	1.0	0	0	0
P_1	log(0.0032)	61.10	0.5	2.35	0.95	0.80
P_2	log(0.01)	76.80	1.0	1.37	0	0.10
P_3	log(0.0224)	78.80	0.5	0.05	0.09	0.12
P_4	log(0.05)	80.80	1.0	0.22	0.16	0.03
P_5	log(0.1118)	85.00	0.5	0	0.01	0.05
P_6	log(0.25)	89.20	1.0	0.02	0.02	0.01
P_7	log(2)	100.00	1.0	0	0	0

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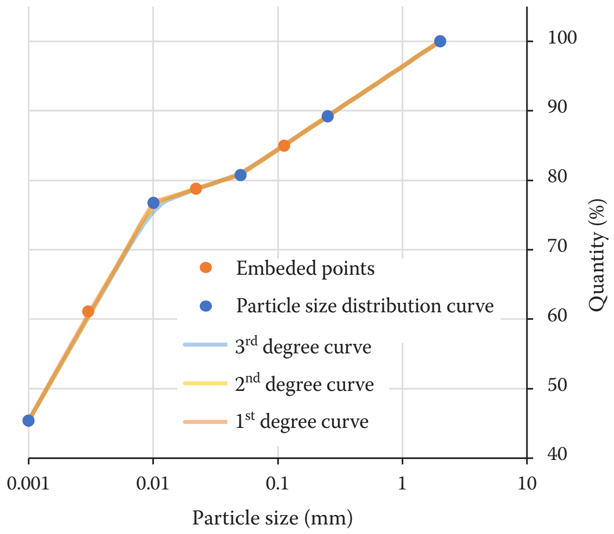

Figure 5. Control points and approximation curves for Example 3 that assumed control points from the particle-size distribution curve in logarithmic scale with the uniform nodal vector of parametrization evaluated using particle-size distribution curve with embedded intermediate values by linear interpolation

approximation curves are also shown in Table S3 in ESM. Differences between the control points and the points of the approximation curves are shown in Table 6. By using the inserted points, we obtain new approximation curves, where the maximum difference between the points of the approximation curves and the points of the particle-size distribution curve for the cubic curve represents 2.35%, for the quadratic curve 0.95%.

Example 4 – Experimentally determined particle-size distribution curve with embedded intermediate values by linear interpolation and non-uniform nodal vector of parametrization determined by center of gravity method

Other options for refining the approximation are the possibility of changing the weight for the individual control points of the approximate polygon, as well as the use of the center of gravity method for calculating the parametrization vector. The calculation of the parametrization vector itself is carried out analogously to the example without embedded points, the particle-size distribution curve given above.

Resulting approximation curves are shown together with the control points in Figure 6. Points of the approximation curves are also shown in Table S4 in ESM. Differences between the control points and the points of the approximation curves are shown in Table 7. By using embedded points and a non-uniform nodal vector of parametrization determined by the center of gravity method, new approximation curves were obtained, where the maximum difference between the points of the approximation curves and the points of the particle-size distribution curve is 1.94% for the cubic curve and 0.86% for the quadratic curve. If it was considered only the original points of the curve, then the maximum difference, for a quadratic curve, is only 0.64%.

Example 5 – Control points from particle-size distribution curve in logarithmic scale with uniform weight

Control points P_i for B particle-size distribution curve and their weights (for simplicity, we consider the same weight for all control points: $w = 1$) are shown in Table 8.

Uniform nodal vector of parametrization

To simplify the example, a uniform nodal vector of parametrization was cosidered.

For the individual degrees of the approximation curve considered (where we proceed from the condition that the degree of the approximation curve

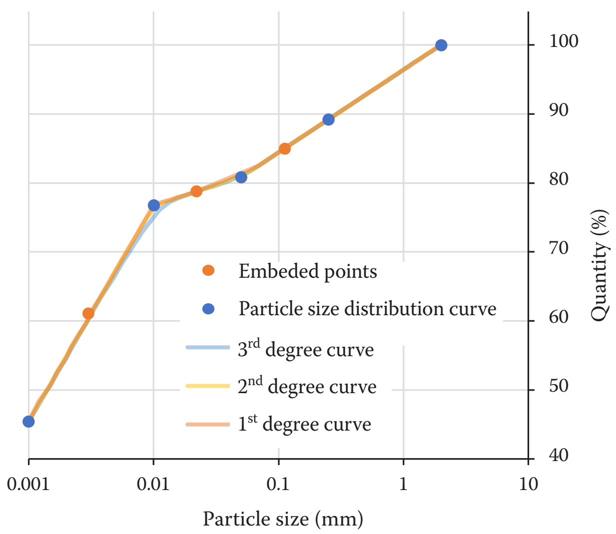

Figure 6. Control points and approximation curves for Example 4 that assumed control points from the particle-size distribution curve in logarithmic scale with the non-uniform nodal vector of parametrization determined by the center of gravity method evaluated using particle-size distribution curve with embedded intermediate values by linear interpolation

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Table 7. Control points (P_i), their weights, and differences between the control points and the points of the approximation curves obtained for Example 4 that assumed the non-uniform nodal vector of parametrization determined by the center of gravity method and particle-size distribution curve with embedded intermediate values for particle diameter d = 0.0032, 0.0224, and 0.1118 mm by linear interpolation

P_i	$X = \log(d)$	Y = cumulative p (%)	Weight	3rd degree curve	2nd degree curve	1st degree curve
					(%)	
P_0	log(0.001)	45.40	1.0	0	0	0
P_1	log(0.0032)	61.10	0.5	1.00	0.86	0.84
P_2	log(0.01)	76.80	1.0	1.94	0.64	0.13
P_3	log(0.0224)	78.80	0.5	0.05	0	0.08
P_4	log(0.05)	80.80	1.0	0.19	0.22	0.16
P_5	log(0.1118)	85.00	0.5	0.04	0.02	0.02
P_6	log(0.25)	89.20	1.0	0.03	0.02	0
P_7	log(2)	100.00	1.0	0	0	0

must be smaller than the number of sections of the approximated polygon) $n < m$, we get the following parametrization vectors:

For a curve of the 1st degree (2nd order) – (linear B-spline) and a sequence of 5 control points:

t = (−2.6990, −2.6990, −1.9490, −1.1990, −0.4490, 0.3010, 0.3010)

For a curve of the 2nd degree (3rd order) – (quadratic B-spline) and a sequence of 5 control points:

t = (−2.6990, −2.6990, −2.6990, −1.6990, −0.6990, 0.3010, 0.3010, 0.3010)

For a curve of the 3rd degree (4th order) – (cubic B-spline) and a sequence of 5 control points:

t = (−2.6990, −2.6990, −2.6990, −2.6990, −1.1990, 0.3010, 0.3010, 0.3010, 0.3010)

Resulting approximation curves are shown together with the control points in Figure 7. Points of the approximation curves are also shown in Table S5 in ESM. Differences between the control points and the points of the approximation curves are shown in Table 8. The difference between the points of the approximation curves and the points of the particle-size distribution curve is therefore 6.94% for the cubic curve, 3.46% for the quadratic curve. The linear function only for comparison was considered.

Example 6 – Control points from particle-size distribution curve in logarithmic scale with uniform weight and nodal vector of parametrization determined by center of gravity method

Nodal vector of parametrization determined by center of gravity method

For a curve of the 1st degree (2nd order) – (linear B-spline) and a sequence of 5 control points:

t = (−2.6990, −2.6990, −2.5559, −1.9162, −0.9171, 0.3010, 0.3010)

For a curve of the 2nd degree (3rd order) – (quadratic B-spline) and a sequence of 5 control points:

Table 8. Control points (P_i), their weights, and differences between the control points and the points of the approximation curves (3rd, 2nd, and 1st degree) obtained for Example 5 that assumed control points from the particle-size distribution curve in logarithmic scale with the uniform nodal vector of parametrization

P_i	$X = \log(d)$	Y = cumulative p (%)	Weight	3rd degree curve	2nd degree curve	1st degree curve
					(%)	
P_0	log(0.002)	22.50	1	0.00	0.00	0.00
P_1	log(0.01)	22.50	1	2.77	1.86	0.45
P_2	log(0.05)	33.40	1	6.94	3.46	0.51
P_3	log(0.25)	72.10	1	−3.65	−3.34	−1.01
P_4	log(2)	100.00	1	0.00	0.00	0.00

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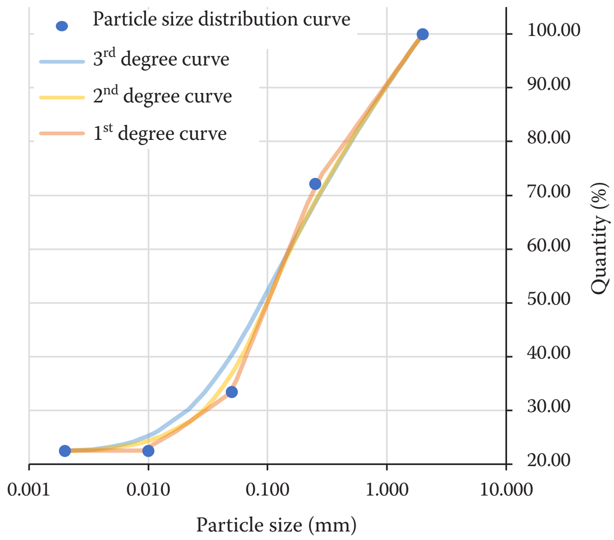

Figure 7. Control points and approximation curves for Example 5 that assumed control points from the particle-size distribution curve in logarithmic scale with the uniform nodal vector of parametrization

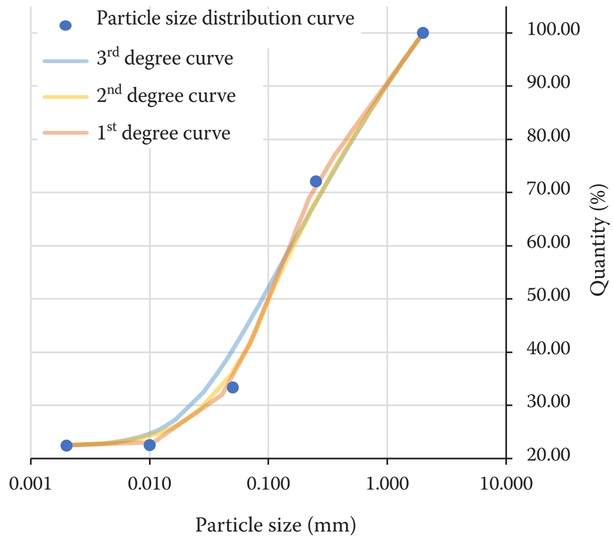

Figure 8. Control points and approximation curves for Example 6 that assumed control points from the particle-size distribution curve in logarithmic scale with the non-uniform nodal vector of parametrization determined by the center of gravity method

t = (−2.6990, −2.6990, −2.6990, −2.1126, −1.3626, 0.3010, 0.3010, 0.3010)

For a curve of the 3rd degree (4th order) – (cubic B-spline) and a sequence of 5 control points:

t = (−2.6990, −2.6990, −2.6990, −2.6990, −1.6300, 0.3010, 0.3010, 0.3010, 0.3010)

Resulting approximation curves are shown together with the control points in Figure 8. Points of the approximation curves are also shown in Table S6 in ESM. Differences between the control points and the points of the approximation curves are shown in Table 9. The difference between the points of the approximation curves and the points of the particle-size distribution curve is therefore 7.03% for the cubic curve, 4.02% for the quadratic curve. We consider the linear function only for comparison.

Example 7 – Inserting of intermediate values in the sequence of control points

For embedded values on the summation line, we use the logarithms of the particle size. Intermediate points of the particle-size distribution curve are determined by linear interpolation. For approximation, we then have a sequence of 8 points of the particle-size distribution curve B (Table 2).

Control points P_i for B particle-size distribution curve and their weights (for simplicity, were considered, for original particle-size distribution curve

Table 9. Control points (P_i), their weights, and differences between the control points and the points of the approximation curves (3rd, 2nd, and 1st degree) obtained for Example 6 that assumed control points from the particle-size distribution curve in logarithmic scale with the non-uniform nodal vector of parametrization determined by the center of gravity method

P_i	$X = \log(d)$	Y = cumulative p (%)	Weight	3rd degree curve	2nd degree curve	1st degree curve
					(%)	
P_0	log(0.002)	22.50	1	0.00	0.00	0.00
P_1	log(0.01)	22.50	1	2.22	1.74	0.51
P_2	log(0.05)	33.40	1	7.03	2.85	1.65
P_3	log(0.25)	72.10	1	−4.07	−4.02	−1.43
P_4	log(2)	100.00	1	0.00	0.00	0.00

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Table 10. Control points (P_i), their weights, and differences between the control points and the points of the approximation curves obtained for Example 7 that assumed the uniform nodal vector of parametrization and particle-size distribution curve with embedded intermediate values for particle diameter d = 0.0045, 0.0224 and 0.1118 mm by linear interpolation

P_i	X = log(d)	Y = cumulative p (%)	Weight	3rd degree curve	2nd degree curve	1st degree curve
					(%)	
P_0	log(0.002)	22.50	1.0	0.00	0.00	0.00
P_1	log(0.0045)	22.50	0.5	–0.05	0.00	0.00
P_2	log(0.01)	22.50	1.0	–0.70	–0.47	–0.54
P_3	log(0.0224)	27.95	0.5	0.03	0.08	0.05
P_4	log(0.05)	33.40	1.0	–1.55	–1.11	–0.17
P_5	log(0.1118)	52.75	0.5	0.14	0.30	0.22
P_6	log(0.25)	72.10	1.0	2.23	1.52	0.27
P_7	log(2)	100.00	1.0	0.00	0.00	0.00

values, uniform weight w = 1 and, for embedded values uniform weight w = 0.5) are presented in Table 10.

Uniform nodal vector of parametrization

To simplify the example, a uniform parametrization vector was first considered.

For the individual degrees of the curve considered (where we proceed from the condition that the degree of the approximation curve must be smaller than the number of sections of the approximated polygon) $n < m$, the following parametrization vectors was get:

For a curve of the 1st degree (2nd order) – (linear B-spline) and a sequence of 8 control points:

$$t = (–2.6990, –2.6990, –2.2704, –1.8418, –1.4133, –0.9847, –0.5561, –0.1275, 0.3010, 0.3010)$$

where the initial 2 nodes correspond to log(0.002), the final 2 nodes correspond to log(2), and the remaining 6 nodes equidistantly divide the intermediate interval.

For a curve of 2nd degree (3rd order) – (quadratic B-spline) and a sequence of 8 control points:

$$t = (–2.6990, –2.6990, –2.6990, –2.19897, –1.6990, –1.1990, –0.6990, –0.1990, 0.3010, 0.3010, 0.3010)$$

where the initial 3 nodes correspond to log(0.002), the final 3 nodes correspond to log(2), and the remaining 5 nodes equidistantly divide the intermediate interval.

For a curve of 3rd degree (4th order) – (cubic B-spline) and a sequence of 8 control points:

$$t = (–2.6990, –2.6990, –2.6990, –2.6990, –2.0990, –1.4990, –0.8990, –0.2990, 0.3010, 0.3010, 0.3010, 0.3010)$$

where the initial 4 nodes correspond to log(0.002), the final 4 nodes correspond to log(2), and the remaining 4 nodes equidistantly divide the intermediate interval.

Resulting approximation curves are shown together with the control points in Figure 9. Points of the approximation curves are also shown in Table S7 in ESM. Differences between the control points and the points of the approximation curves are shown in Table 10. By using the embedded points, a new approximation curves was obtained, where the maximum difference between

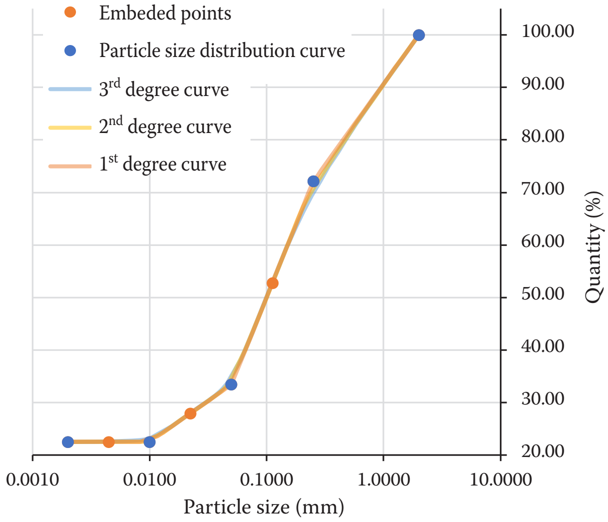

Figure 9. Control points and approximation curves for Example 7 that assumed control points from the particle-size distribution curve in logarithmic scale with the uniform nodal vector of parametrization evaluated using particle-size distribution curve with embedded intermediate values by linear interpolation

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Table 11. Control points (P_i), their weights, and differences between the control points and the points of the approximation curves obtained for Example 8 that assumed the non-uniform nodal vector of parametrization and particle-size distribution curve with embedded intermediate values for particle diameter d = 0.0045, 0.0224 and 0.1118 mm by linear interpolation

P_i	X = log(d)	Y = cumulative p (%)	Weight	3rd degree curve (%)	2nd degree curve (%)	1st degree curve (%)
P_0	log(0.002)	22.50	1.0	0.00	0.00	0.00
P_1	log(0.0045)	22.50	0.5	0.35	–0.28	0.00
P_2	log(0.01)	22.50	1.0	–0.63	–0.89	–0.54
P_3	log(0.0224)	27.95	0.5	0.06	0.14	0.05
P_4	log(0.05)	33.40	1.0	–1.34	–1.06	–0.17
P_5	log(0.1118)	52.75	0.5	0.20	0.07	0.22
P_6	log(0.25)	72.10	1.0	3.13	1.93	0.27
P_7	log(2)	100.00	1.0	0.00	0.00	0.00

the points of the approximation curves and the points of the particle-size distribution curve for the cubic curve represents 1.55% and for the quadratic curve 1.52%.

Example 8 – Experimentally determined particle-size distribution curve with embedded intermediate values by linear interpolation and non-uniform nodal vector of parametrization determined by center of gravity method

In this example is determined, for refining the approximation, non-uniform nodal vector of parametrization by center of gravity method. The calculation of the parametrization vector itself is carried out analogously to the example without embedded points given above (Example 2). For approximation, there was a sequence of 8 points of the particle-size distribution curve B (Table 2).

Control points P_i for B particle-size distribution curve and their weights (for simplicity, it was considered, for original particle-size distribution curve values, uniform weight w = 1 and, for embedded values, uniform weight w = 0.5) are presented in Table 11.

Non uniform nodal vector of parametrization determined by center of gravity method

For a curve of the 1st degree (2nd order) – (linear B-spline) and a sequence of 8 control points:

t = (–2.6990, –2.6990, –2.6855, –2.6143, –2.4737, –2.0853, –1.5186, –0.6637, 0.3010, 0.3010)

For a curve of the 2nd degree (3rd order) – (quadratic B-spline) and a sequence of 8 control points:

t = (–2.6990, –2.6990, –2.6990, –2.6425, –2.5364, –2.2439, –1.7646, –1.0685, 0.3010, 0.3010, 0.3010)

For a curve of the 3rd degree (4th order) – (cubic B-spline) and a sequence of 8 control points:

t = (–2.6990, –2.6990, –2.6990, –2.6990, –2.5697, –2.3354, –1.9517, –1.3524, 0.3010, 0.3010, 0.3010, 0.3010)

Resulting approximation curves are shown together with the control points in Figure 10. Points of the approximation curves are also shown in Table S8

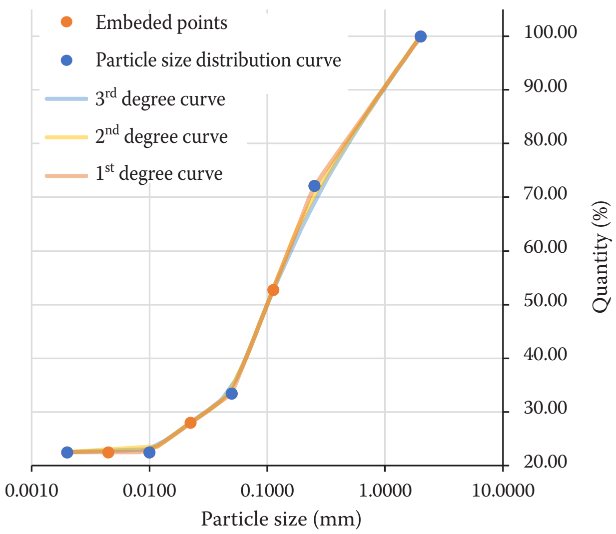

Figure 10. Control points and approximation curves for Example 8 that assumed control points from the particle-size distribution curve in logarithmic scale with the non-uniform nodal vector of parametrization determined by center of gravity method, evaluated using particle-size distribution curve with embedded intermediate values by linear interpolation

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in ESM. Differences between the control points and the points of the approximation curves are shown in Table 11. By using the embedded points, we obtain new approximation curves, where the maximum difference between the points of the approximation curves and the points of particle-size distribution curve for the cubic curve represents 3.13% and for the quadratic curve 1.93%.

It could be concluded that for the approximation of particle-size distribution curve was chosen approximative NURBS curve. It brings, into standard B-spline curve, rational elements in the form of weights of each single controlling point and aspect of non-uniformity in the form of nod vector of parametrization. In contrast to standard B-spline curve it makes it possible through weights of single controlling points to influence the final shape of approximation. Together with node vector of parametrization to determine location of each point, where there are joint single segments of approximation curve. It is necessary to mention key features of approximation curve, where polygon formed by sequence of check points do not correspond with a standard texture curve with experimentally determined breaking points and is forming convex envelope of the final NURBS curve. It means that also the final NURBS curve respects character of particle size cumulative curve.

It was found that NURBS curves represent the key generalization of B-spline curves. It could make it possible to construct particle size cumulative curves with emphasis on all its variants and restrain a crucial shortage of approximation represented by lack of control points.

CONCLUSION

Approximation of particle-size distribution curves by means of B-spline curves, or their generalized NURBS variant, using weights for individual points of the curve and a non-uniform nodal vector of parametrization representing the division of the approximate polygon of the particle-size distribution curve into sub-polygons, enables the derivation of a smooth and continuous curve. However, the construction of the approximation curve is primarily dependent on the limited number of points of the experimentally determined particle-size distribution curves, and for curves with significant breaks in the course, one must consider either a lower accuracy of the approximation or the necessity of "improving" the approximation using the weights of individual points, inserting additional points or working with a nodal vector of parametrization. For basic approximation, the PUGIS system offers automatic approximation using all variants mentioned in the text, as well as the possibility of individual changes in the weights of control points, in their number and position, and in the nodal vector of parametrization.

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